Recall Let f be a complex-valued function such that f is not holomorphic at z_0 and f is holomorphic on some open disk $\{z \in \mathbb{C} : |z - z_0| < R\}$. Then we call z_0 an isolated singularity of f.

Let z_0 be an isolated singularity of f. Then f is holomorphic on $\{z \in \mathbb{C} : |z - z_0| < R\}$ for some $R > 0$. So,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}, \quad 0 < |z - z_0| < R.$$

Case 1: $a_{-n} = 0$, $n = 1, 2, \cdots \Rightarrow z_0$ is ~~[illegible]~~ a removable singularity.

Case 2: $a_{-m} \neq 0$ for some $m \in \mathbb{N}$ and $a_{-n} = 0$ for all $n > m \Rightarrow z_0$ is a pole of order m. If $m = 1$, then we call z_0 a simple pole.

Case 3: $a_{-n} \neq 0$ for infinitely many n in $\mathbb{N} \Rightarrow z_0$ is an essential singularity.

Example 1: Let $f(z) = e^{1/z}$. Find all isolated singularities. What kind of singularity is for each one?

Solution 0 is the only ~~a~~ isolated singularity. The Laurent series of f is

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} = 1 + \sum_{n=1}^{\infty} a_{-n} z^{-n}, \quad 0 < |z| < \infty,$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} z^{-n}, \quad 0 < |z| < \infty.$$

$\therefore a_{-n} = \frac{1}{n!}$, $n = 1, 2, \cdots$.

$\therefore 0$ is an essential singularity.

Example 2: Repeat Example 1 for

(a) $f(z) = \dfrac{\sin z}{z}$,

(b) $f(z) = \dfrac{e^z}{z^2}$.

Solution: (a) 0 is the only isolated singularity.

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

$\therefore a_{-n} = 0$ for all $n = 1, 2, \cdots$. $\therefore$ 0 is a removable singularity. (b) 0 is the only singularity.

$$\frac{e^z}{z^2} = \frac{1}{z^2}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)$$

$$= \left(\frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{3!} + \cdots\right) + \left(z^{-1} + z^{-2}\right).$$

$\therefore a_{-2} = 1 \neq 0$ and $a_{-n} = 0$ for $n > 2$. $\therefore$ 0 is a pole of order 2.

Definition: Let z_0 be an isolated singularity of a complex valued function $w = f(z)$. Then f is holomorphic on a punctured disk $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Expand f in Laurent series as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}, \quad 0 < |z - z_0| < R.$$

Call a_{-1} the residue of f at z_0 and we write

$$a_{-1} = \operatorname{Res}(f, z_0).$$

Example: Let z_0 be a removable singularity of f. Compute $\operatorname{Res}(f, z_0)$.

Solution: $a_{-1} = 0 \Rightarrow \operatorname{Res}(f, z_0) = 0$.

Example Let z_0 be a simple pole of f. Compute $\operatorname{Res}(f, z_0)$.

Solution Write f in its Laurent series as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + a_{-1}(z - z_0)^{-1}, \quad 0 < |z - z_0| < R.$$

$$\therefore\ (z-z_0)f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+1} + a_{-1}$$

$$\therefore\ \lim_{z\to z_0}(z-z_0)f(z) = a_{-1} \Rightarrow \mathrm{Res}(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z).$$

Example: Let z_0 be a pole of order m of f. Compute $\mathrm{Res}(f, z_0)$.

Solution: Write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} a_{-n}(z-z_0)^{-n}, \quad 0<|z-z_0|<R.$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \left\{ a_{-1}(z-z_0)^{-1} + a_{-2}(z-z_0)^{-2} + \cdots + a_{-m}(z-z_0)^{-m} \right\}$$

$$\therefore\ (z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + \left\{ a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-2}(z-z_0)^{m-2} + a_{-1}(z-z_0)^{m-1} \right\}$$

Recall

$$\frac{d^\alpha}{dx^\alpha}\left(x^\beta\right) = \begin{cases} 0 & \text{if } \alpha>\beta \\ \beta(\beta-1)\cdots(\beta-\alpha+1)\,x^{\beta-\alpha}, & \alpha\le\beta \end{cases}$$

$$= \begin{cases} 0 & \text{if } \alpha>\beta \\ \dfrac{\beta!}{(\beta-\alpha)!}\,x^{\beta-\alpha}, & \alpha\le\beta \end{cases} = \begin{cases} 0 & \text{if } \alpha>\beta, \\ \binom{\beta}{\alpha}\alpha!\,x^{\beta-\alpha}, & \alpha\le\beta \end{cases}$$

$$\therefore\ \left(\frac{d}{dz}\right)^{m-1}\left\{(z-z_0)^m f(z)\right\} = \sum_{n=0}^{\infty} a_n \binom{n+m}{m-1}(m-1)!\,(z-z_0)^{n+1} + a_{-1}$$

$$\therefore\ \mathrm{Res}(f, z_0) = a_{-1} = \lim_{z\to z_0}\left(\frac{d}{dz}\right)^{m-1}\left\{(z-z_0)^m f(z)\right\}.$$

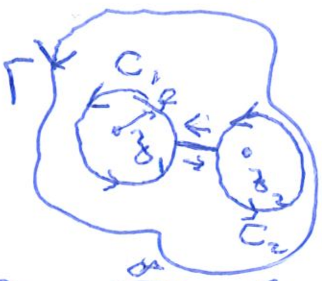
Why Residues?

Theorem (Cauchy's Residue Theorem)

Let Γ be a simple closed contour oriented once in the counterclockwise direction. Let $w = f(z)$ be a holomorphic function on and inside Γ except at the isolated singularities $z_1, \dots, z_N$ inside Γ.

Then $$\int_\Gamma f(z)\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}(f, z_j).$$

Proof: ($N = 2$ only)



$\therefore$ by Cauchy's Integral Theorem,

$$\int_\Gamma f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_1)^{-n}, \quad 0 < |z - z_1| < R$$

$$\therefore \int_{C_1} f(z)\,dz = \sum_{n=0}^{\infty} a_n \int_{C_1} (z - z_0)^n\,dz + \sum_{n=1}^{\infty} a_{-n} \int_{C_1} (z - z_1)^{-n}\,dz$$

$$= 0 + 2\pi i\, a_{-1} = 2\pi i \operatorname{Res}(f, z_1)$$

Similarly,

$$\int_{C_2} f(z)\,dz = 2\pi i \operatorname{Res}(f, z_2).$$

$$\therefore \int_\Gamma f(z)\,dz = 2\pi i \left(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2) \right).$$